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Quadratic forms associated to stratifying systems ☆

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Abstract

Let R be an algebra, and let $(\theta, \preccurlyeq)$ be a stratifying system of R-modules. If the category $\mathcal{F}(\theta)$ is θ-directing, then we prove that $\operatorname{ind}\mathcal{F}(\theta)$ is finite. In order to do that, we introduce a quadratic form q_θ which depends on θ. Moreover, we also give sufficient conditions to get the correspondence $X \mapsto \underline{\dim}_\theta X$ from $\operatorname{ind}\mathcal{F}(\theta)$ to the set of positive roots of q_θ.

Keywords: Standardly stratified algebras; Quadratic forms; Stratifying systems

0. Introduction

In this paper, algebra means finite-dimensional basic algebra over an algebraically closed field k. If R is an algebra, the category of finitely generated left R-modules is denoted by $\operatorname{mod} R$, and the usual duality $\operatorname{Hom}_k(-, k) : \operatorname{mod} R \to \operatorname{mod} R^{\mathrm{op}}$ is denoted by D. All the subcategories of

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mod R to be considered are full subcategories. Given $f: M \to N$ and $g: N \to L$ morphisms in mod R we denote the composition of f and g by gf, which is a morphism from M to L.

Given a class $\mathcal{C}$ of R-modules, we denote by $\mathcal{F}(\mathcal{C})$ the subcategory of mod R whose objects are the zero module and all modules which are filtered by modules in $\mathcal{C}$. That is, a non-zero R-module M belongs to $\mathcal{F}(\mathcal{C})$ if there is a finite chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$$

of submodules of M such that M_i/M_{i-1} is isomorphic to a module in $\mathcal{C}$ for all $i = 1, 2, \ldots, m$. In fact, $\mathcal{F}(\mathcal{C})$ is closed under extensions and $\mathcal{F}(\emptyset) = \{0\}$.

Let R be an algebra and $\{\varepsilon_1, \ldots, \varepsilon_s\}$ be a complete set of primitive orthogonal idempotents. Then, we fix the natural order on the set of indices $[1, s] = \{1, \ldots, s\}$. Let $P(i) = R\varepsilon_i$ be the indecomposable projective R-module corresponding to the idempotent ε_i, and $S(i)$ be the simple top $P(i)/\operatorname{rad} P(i)$ of $P(i)$ for $1 \leqslant i \leqslant s$. The *standard* module ${}_R\Delta(i)$ is, by definition, the maximal factor module of $P(i)$ without composition factors $S(j)$ for $j > i$. We denote by ${}_R\Delta$ the set $\{{}_R\Delta(i)\}_{i\in[1,s]}$.

We recall that an algebra R, with a fixed total order of the simple modules, is called *standardly stratified* if ${}_RR \in \mathcal{F}({}_R\Delta)$. A standardly stratified algebra R is called *quasi-hereditary* if $\dim_k \operatorname{End}({}_R\Delta(i)) = 1$ for any i. Quasi-hereditary algebras were introduced by E. Cline, B.J. Parshall and L.L. Scott in [3] and standardly stratified algebras by V. Dlab in [5]; furthermore, a new theory of stratified algebras has been given in [4].

Instead of a partial order on the iso-classes of simple modules, as was done in [3], we will consider only total orders. In addition, typical examples of quasi-hereditary algebras are hereditary algebras. In fact, it was proved by V. Dlab and C.M. Ringel in [6] that an algebra is hereditary if and only if it is a quasi-hereditary algebra for any total order of the simple modules.

Quadratic forms play an important role in the representation theory of finite-dimensional algebras. Thus, it makes sense to try them also for "relative representation theory," that is, for certain subcategories of modules. In the context of quasi-hereditary algebras, S. Liu and C. Xi considered in [9] hereditary algebras as quasi-hereditary algebras. They introduced a quadratic form $q_{{}_R\Delta}$ for a quasi-hereditary algebra R, and proved that if R is a hereditary algebra, then $\mathcal{F}({}_R\Delta)$ is finite if and only if $q_{{}_R\Delta}$ is weakly positive. Later on, in [2], B. Deng studied the category $\mathcal{F}({}_R\Delta)$ with R a quasi-hereditary algebra. Analogously as it is usually done for mod R, B. Deng studied ${}_R\Delta$-directing and ${}_R\Delta$-omnipresent modules in $\mathcal{F}({}_R\Delta)$, and proved that the existence of a ${}_R\Delta$-directing and ${}_R\Delta$-omnipresent module in $\mathcal{F}({}_R\Delta)$ implies that all standard modules have projective dimension at most 2. Afterwards, by using the process of standardization introduced by V. Dlab and C.M. Ringel in [7], B. Deng showed that the study of ${}_R\Delta$-directing modules in $\mathcal{F}({}_R\Delta)$ can be reduced to the study of those over certain quasi-hereditary algebra B which admits a ${}_B\Delta$-directing and ${}_B\Delta$-omnipresent module. Moreover, B. Deng proved that, for the algebras R and B, the quadratic forms associated to their categories $\mathcal{F}(\Delta)$ are essentially the same. Then, using this reduction, B. Deng proved that $\mathcal{F}({}_R\Delta)$ is finite if all the indecomposable modules in $\mathcal{F}({}_R\Delta)$ are ${}_R\Delta$-directing.

The generalization of those results, obtained by B. Deng, to standardly stratified algebras are not so obviously. First, B. Deng used the fact that the Cartan matrix of a quasi-hereditary algebra R is invertible, and it is the case since R has finite global dimension; however, in general, the global dimension of a standardly stratified algebra is infinity. Second, B. Deng uses the "Process of standardization" given by V. Dlab and C.M. Ringel in [7] which is only valid for quasi-hereditary algebras; in the case of standardly stratified algebras, we have to use a more generalized "process of standardization."

So, we have to look for a more generalized "process of standardization" for standardly stratified algebras. Fortunately, that process exists, and that is why the stratifying systems appears in this work. On the one hand, the stratifying systems generalizes the "Process of standardization" given by V. Dlab and C.M. Ringel in [7] (see Theorem 3.2 in [14]). On the other hand, stratifying systems generalizes the standard modules ${}_R\Delta$. Then, we go further and consider the problem directly for stratifying systems. One of the main result in this paper is the following: "if $(\theta, \preccurlyeq)$ is a stratifying system and $\mathcal{F}(\theta)$ is θ-directing, then $\mathcal{F}(\theta)$ is of finite representation type."

In the paper, we consider the quadratic form q_θ associated to a stratifying system $(\theta, \preccurlyeq)$. Later, we study, for standardly stratified algebras, ${}_R\Delta$-directing and ${}_R\Delta$-omnipresent modules in the category $\mathcal{F}({}_R\Delta)$, and show that the existence of one of such modules X implies that all modules in $\mathcal{F}({}_R\Delta)$ have projective dimension at most 2, and also that X has projective (respectively relative injective) dimension at most 1. On the other hand, using the Ext-projective stratifying systems $(\theta, \underline{Q}, \preccurlyeq)$ introduced in [14], we show that the study of θ-directing modules in $\mathcal{F}(\theta)$ can be reduced to the study of those modules over certain standardly stratified algebra B that admits a ${}_B\Delta$-directing and ${}_B\Delta$-omnipresent module. Moreover, we also prove that the quadratic forms q_θ and $q_{{}_B\Delta}$ are essentially the same. Finally, using this reduction, we prove that $\operatorname{ind}\mathcal{F}(\theta)$ is finite if all the indecomposable modules in $\mathcal{F}(\theta)$ are θ-directing.

1. Preliminaries

Throughout the paper, we denote by $[1, t]$ the set $\{1, 2, \ldots, t\}$ and by $\preccurlyeq$ a total order on $[1, t]$. However, we reserve the notation $\leqslant$ for the natural order on $[1, t]$.

Let R be an algebra. We start this section by recalling the definition of stratifying system, Ext-injective stratifying system and Ext-projective stratifying system given in [13,14]. Afterwards, we recall the notion of standard stratifying system. Finally, we introduce briefly the notion of relative projective dimension, $\mathcal{X}$-resolution dimension and Euler's quadratic form.

Definition 1.1. [13] A stratifying system $(\theta, \preccurlyeq)$ of size t consists of a set $\theta = \{\theta(i)\}_{i=1}^t$ of indecomposable R-modules and a total order $\preccurlyeq$ on $[1, t]$, satisfying the following conditions:

(a) $\operatorname{Hom}_R(\theta(j), \theta(i)) = 0$ for $j \succ i$,
(b) $\operatorname{Ext}_R^1(\theta(j), \theta(i)) = 0$ for $j \succcurlyeq i$.

In the theory of stratifying systems, there are three equivalent notions: (a) Stratifying systems (see 1.1), (b) Ext-injective stratifying system (see 1.2), and (c) Ext-projective stratifying system (see 1.3). The equivalence of those notions implies in particular that, given a stratifying system $(\theta, \preccurlyeq)$ of size t, we can associate to it an uniquely determined Ext-injective stratifying system (*eiss* for short) $(\theta, \underline{Y}, \preccurlyeq)$ and an uniquely determined Ext-projective stratifying system (*epss* for short) $(\theta, \underline{Q}, \preccurlyeq)$. Moreover, the set $\underline{Y} = \{Y(1), \ldots, Y(t)\}$ (respectively $\underline{Q} = \{Q(1), \ldots, Q(t)\}$) consists of pairwise non-isomorphic indecomposable R-modules. In order to simplify the statements, we set $Y = \coprod_{i=1}^t Y(i)$ and $Q = \coprod_{i=1}^t Q(i)$. Finally, we introduce the categories $\mathcal{P}(\theta)$ and $\mathcal{I}(\theta)$, which will be used throughout the paper

$$\mathcal{P}(\theta)=\big\{X\in \operatorname{mod} R\colon\ \operatorname{Ext}^1_R(X.-)\big|_{\mathcal{F}(\theta)}=0\big\}\quad \text{and}$$
$$\mathcal{I}(\theta)=\big\{X\in \operatorname{mod} R\colon\ \operatorname{Ext}^1_R(-,X)\big|_{\mathcal{F}(\theta)}=0\big\}.$$

Definition 1.2. [8] Let $\theta=\{\theta(i)\}_{i=1}^t$ be a set of non-zero R-modules, $\underline{Y}=\{Y_i\}_{i=1}^t$ a set of indecomposable R-modules, and $\preccurlyeq$ a total order on $[1,t]$. The triple $(\theta,\underline{Y},\preccurlyeq)$ is an Ext-injective stratifying system of size t if the following three conditions hold:

(a) $\operatorname{Hom}_R(\theta(j),\theta(i))=0$ for $j\succ i$,
(b) for each $i\in[1,t]$, there is an exact sequence $0\to\theta(i)\xrightarrow{\alpha_i}Y(i)\to Z(i)\to 0$ such that $Z(i)\in\mathcal{F}(\{\theta(j)\colon\ j\prec i\})$,
(c) $\operatorname{Ext}^1_R(-,Y)|_{\mathcal{F}(\theta)}=0$.

Definition 1.3. [14] Let $\theta=\{\theta(i)\}_{i=1}^t$ be a set of non-zero R-modules, $\underline{Q}=\{Q(i)\}_{i=1}^t$ a set of indecomposable R-modules and $\preccurlyeq$ a total order on $[1,t]$. The triple $(\theta,\underline{Q},\preccurlyeq)$ is an Ext-projective stratifying system of size t if the following three conditions hold:

(a) $\operatorname{Hom}_R(\theta(j),\theta(i))=0$ for $j\succ i$,
(b) for each $i\in[1,t]$, there is an exact sequence $0\to K(i)\to Q(i)\xrightarrow{\beta_i}\theta(i)\to 0$ such that $K(i)\in\mathcal{F}(\{\theta(j)\colon\ j\succ i\})$,
(c) $\operatorname{Ext}^1_R(Q,-)|_{\mathcal{F}(\theta)}=0$.

Let $(\theta,\preccurlyeq)$ be a stratifying system. K. Erdmann and C. Saenz have shown in [8] that the filtration multiplicities $[M:\theta(i)]$ do not depend on the filtration of $M\in\mathcal{F}(\theta)$. For this reason, we can introduce the θ-support of M as the set $\operatorname{Supp}_\theta(M)=\{i\in[1,t]\colon\ [M:\theta(i)]\neq 0\}$. Therefore, $\operatorname{Supp}_\theta(M)$ is empty if $M=0$. We define the functions $\min,\max\colon\mathcal{F}(\theta)\to[1,t]\cup\{\pm\infty\}$ as follows: (a) $\min(0):=+\infty$ and $\max(0):=-\infty$, and (b) $\min(M):=\min(\operatorname{Supp}_\theta(M),\preccurlyeq)$ and $\max(M):=\max(\operatorname{Supp}_\theta(M),\preccurlyeq)$ if $M\neq 0$. Finally, we recall that a stratifying system $(\theta,\preccurlyeq)$ of size t is *standard* if ${}_RR\in\mathcal{F}(\theta)$.

Let $\mathcal{X}$ be a class of R-modules. We denote by $\mathcal{X}^\wedge$ the subcategory of mod R whose objects are those R-modules X for which there exists a finite $\mathcal{X}$-resolution. That is, $M\in\mathcal{X}^\wedge$ if and only if there exists a long exact sequence $0\to X_n\to\cdots\to X_1\to X_0\to M\to 0$ with $X_i\in\mathcal{X}$ for all $i=0,1,\dots,n$. Dually, $\mathcal{X}^\vee$ is the subcategory of mod R whose objects have a finite $\mathcal{X}$-coresolution. We denote by $\operatorname{pd} X$ the projective dimension of X. Similarly, we use the notation $\operatorname{id} X$ for the injective dimension of X. Following Auslander and Buchweitz in [1], we recall the concepts of relative projective dimension and relative resolution dimension of a given module.

Definition 1.4. Let $\mathcal{X}$ be a class of objects in mod R, and M be an R-module.

(a) We shall denote by $\operatorname{pd}_{\mathcal{X}} M$ the *relative projective dimension* of M with respect to $\mathcal{X}$. That is, $\operatorname{pd}_{\mathcal{X}} M:=-\infty$ if $M=0$, and $\operatorname{pd}_{\mathcal{X}} M:=\min\{n\colon\ \operatorname{Ext}^j_R(M,-)|_{\mathcal{X}}=0$ for any $j>n\}$ if $M\neq 0$. Dually, $\operatorname{id}_{\mathcal{X}} M$ is the *relative injective dimension* of M with respect to $\mathcal{X}$.
(b) We shall denote by $\operatorname{resdim}_{\mathcal{X}} M$ the *$\mathcal{X}$-resolution dimension* of M. That is, $\operatorname{resdim}_{\mathcal{X}} M:=-\infty$ if $M=0$, $\operatorname{resdim}_{\mathcal{X}} M:=+\infty$ if $M\notin\mathcal{X}^\wedge$, and $\operatorname{resdim}_{\mathcal{X}} M:=\min\{r\colon$ there is an exact sequence $0\to X_r\to\cdots\to X_0\to M\to 0$, with $X_i\in\mathcal{X}$ for any $i\}$ if $M\in\mathcal{X}^\wedge$. Dually, we denote by $\operatorname{coresdim}_{\mathcal{X}} M$ the *$\mathcal{X}$-coresolution dimension* of M.

(c) For any class $\mathcal{C}$ of R-modules, we set $\mathrm{pd}_{\mathcal{X}}\,\mathcal{C} := \sup\{\mathrm{pd}_{\mathcal{X}}\,M\colon\ M \in \mathcal{C}\}$ and $\mathrm{resdim}_{\mathcal{X}}\,\mathcal{C} := \sup\{\mathrm{resdim}_{\mathcal{X}}\,M\colon\ M \in \mathcal{C}\}$.

Let A be an algebra, and s be the number of iso-classes of simple A-modules. Let $F(A)$ be the free abelian group with basis the set of isomorphism classes in mod A, and $R(A)$ be the subgroup generated by the formal sums $M' - M + M''$ for any exact sequence $0 \to M' \to M \to M'' \to 0$ in mod A. By definition, the Grothendieck group $K_0(A)$ of A is the quotient $F(A)/R(A)$. By the Jordan–Holder's theorem, it is known that $K_0(R)$ is a free abelian group with basis $e(i) = \underline{\dim}\, S(i)$, where $S(1), S(2), \dots, S(s)$ are the iso-classes of simple A-modules. Using this basis, we may identify $K_0(A)$ with $\mathbb{Z}^s$. Then, we can associate to each A-module M the dimension vector $\underline{\dim}\, M = \sum_{i=1}^{s}[M : S(i)]e(i) \in \mathbb{Z}^s$. We will denote by C_A the *Cartan matrix* of A, which is a $s \times s$ matrix with ij entry equals to $\dim_k \mathrm{Hom}_A(P(i), P(j))$. Thus, the jth column is given by $\underline{\dim}\, P(j)^T$. If C_A is invertible, then C_A^{-T} defines a bilinear form $\langle -,- \rangle_A$ on $K_0(A, \mathbb{Q}) := K_0(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ given by $\langle x, y\rangle_A := x C_A^{-T} y^T$. This bilinear form has the following homological interpretation.

Lemma 1.5. [12] *Assume that the Cartan matrix C_A is invertible. If* $\mathrm{pd}\, M < \infty$ *or* $\mathrm{id}\, N < \infty$ *then*

$$\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle_A = \sum_{t \geqslant 0} (-1)^t \dim_k \mathrm{Ext}_A^t(M, N).$$

If the Cartan matrix of A is invertible, then we have the *Euler's quadratic form* $\chi_A(x) := \langle x, x\rangle_A$ on $K_0(A, \mathbb{Q})$. Hence, if $\mathrm{pd}\, M < \infty$ or $\mathrm{id}\, M < \infty$ then

$$\chi_A(\underline{\dim}\, M) = \sum_{t \geqslant 0} (-1)^t \dim_k \mathrm{Ext}_A^t(M, M).$$

2. Quadratic forms on $K_0(\mathcal{F}(\theta))$

Let R be an algebra, and $(\theta, \preccurlyeq)$ be a stratifying system of size t. Let $F(\mathcal{F}(\theta))$ be the free abelian group with basis the set of isomorphism classes of objects in $\mathcal{F}(\theta)$, and $R(\mathcal{F}(\theta))$ be the subgroup generated by the formal sums $M' - M + M''$ for any exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{F}(\theta)$. We have by definition that the Grothendieck group $K_0(\mathcal{F}(\theta))$ of $\mathcal{F}(\theta)$ is the quotient $F(\mathcal{F}(\theta))/R(\mathcal{F}(\theta))$.

Lemma 2.1. *$K_0(\mathcal{F}(\theta))$ is a free abelian group of rank t with basis the images $[\theta(i)]$ of $\theta(i)$ under the canonical map $F(\mathcal{F}(\theta)) \to K_0(\mathcal{F}(\theta))$.*

Proof. It follows from the fact that for any $M \in \mathcal{F}(\theta)$ the filtration multiplicities $[M : \theta(i)]$ do not depend on a given filtration of M in θ (see [8]). $\square$

Remark 2.2. As we have seen above, the relative simple modules θ in $\mathcal{F}(\theta)$ generate the free group $K_0(\mathcal{F}(\theta))$ because of the fact that the filtration multiplicities $[M : \theta(i)]$ do not depend on a given filtration of M in θ. Moreover, we know that $\mathcal{F}(\theta)$ is a functorially finite subcategory of mod R. However, it is not true that for any functorially finite subcategory $\mathcal{C}$ of mod R we have that the relative simple modules have a "good behavior" as above. Indeed, let R be the hereditary path algebra given by the following quiver

$$\begin{tikzcd} 1 \arrow[r, "\theta"] \arrow[d, "\rho"'] & 3 \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \\ 2 \arrow[r, "\xi"'] & 4 \end{tikzcd} .$$

Consider the R-modules: $M := P(1)/\langle \xi\rho - \alpha\theta \rangle$, $K_1 := P(1)/\langle \xi\rho, \alpha\theta \rangle$ and $K_2 := P(1)/\langle \xi\rho - \alpha\theta, \xi\rho - \beta\theta \rangle$. Let $\mathcal{C} := \operatorname{add}\{S(4), M, K_1, K_2\}$. We have that the relative simple modules in $\mathcal{C}$ are $S(4)$, K_1 and K_2 since the only proper submodule of K_1, K_2 and M lying in $\mathcal{C}$ is $S(4)$ and the quotients $K_1/S(4)$, $K_2/S(4)$ do not belong to $\mathcal{C}$. On the other hand, we have two exact sequences $0 \to S(4) \to M \to K_i \to 0$ for $i = 1, 2$. Since $K_1 \not\simeq K_2$, we get that M admits two different filtrations; and so there is no unicity of filtrations of M with the relative simple modules in $\mathcal{C}$.

Using the basis $\underline{\dim}_\theta\, \theta(i) := [\theta(i)]$, we may identify $K_0(\mathcal{F}(\theta))$ with $\mathbb{Z}^t$. Hence, for any $M \in \mathcal{F}(\theta)$, we have the dimension vector

$$\underline{\dim}_\theta\, M := \sum_{i=1}^{t} \big[M : \theta(i)\big]\, \underline{\dim}_\theta\, \theta(i) \in \mathbb{Z}^t.$$

Definition 2.3. Associated to a stratifying system $(\theta, \preccurlyeq)$ of size t, we have two quadratic forms:

(a) the Tits form $q_\theta : \mathbb{Z}^t \to \mathbb{Z}$ defined by the equality

$$q_\theta(x) := \sum_{l=0}^{2} \sum_{i,j} (-1)^l \dim_k \operatorname{Ext}_R^l\big(\theta(i), \theta(j)\big) x(i) x(j),$$

(b) if $\operatorname{pd}_{\mathcal{F}(\theta)} \theta < \infty$, we have a bilinear form $\langle -,- \rangle_\theta$ on $\mathbb{Z}^t$ defined by

$$\langle x, y \rangle_\theta := \sum_{l \geqslant 0} \sum_{i,j} (-1)^l \dim_k \operatorname{Ext}_R^l\big(\theta(i), \theta(j)\big) x(i) y(j).$$

In this case, the Euler quadratic form $\chi_\theta : \mathbb{Z}^t \to \mathbb{Z}$ is by definition

$$\chi_\theta(x) := \langle x, x \rangle_\theta.$$

In 2.3(b), we needed $\operatorname{pd}_{\mathcal{F}(\theta)} \theta$ to be finite. So, it would be useful to have a condition on θ to get a bound for $\operatorname{pd}_{\mathcal{F}(\theta)} \theta$. The following theorem establishes a bound for this number in terms of the category $\mathcal{I}(\theta)$. We recall that, a given subcategory $\mathcal{X}$ of mod R is called *coresolving* if $\mathcal{X}$ satisfies the following three conditions: (a) closed under extensions, (b) closed under cokernels of injections and (c) contains the injective R-modules. Observe that conditions (a) and (c) are valid for $\mathcal{X} = \mathcal{I}(\theta)$.

Theorem 2.4. [16] *Let $(\theta, \preccurlyeq)$ be a stratifying system of size t, and s be the number of iso-classes of simple R-modules. If $\mathcal{I}(\theta)$ is coresolving then*

$$\operatorname{pd}\mathcal{F}(\theta) \leqslant t \leqslant s \quad \textit{and} \quad \operatorname{pd}_{\mathcal{F}(\theta)}\mathcal{F}(\theta) \leqslant t-1.$$

Let $(\theta, \leqslant)$ be a stratifying system of R-modules of size t, and $(\theta, \underline{Q}, \leqslant)$ be the epss associated to $(\theta, \leqslant)$. We introduce the $t \times t$ matrices D, C_Q and D_Q as follows: (a) $D = (d_{ij})$, where $d_{ij} := \dim_k \operatorname{Hom}_R(Q(i), \theta(j))$, (b) the ij entry of C_Q is equal to $\dim_k \operatorname{Hom}_R(Q(i), Q(j))$, and (c) the jth column of D_Q is given by $\underline{\dim}_\theta\, Q(j)^T$. We recall that the functor $e_Q := \operatorname{Hom}_R(Q, -) : \mathcal{F}(\theta) \to \mathcal{F}({}_B\Delta)$ is an equivalence of exact categories, where $B := \operatorname{End}({}_RQ)^{\mathrm{op}}$ is a standardly stratified algebra and $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t\}$ is a fixed complete set of primitive orthogonal idempotents of B such that $B\varepsilon_i \simeq e_Q(Q(i))$ for all i (see Theorem 3.2 in [14]).

Lemma 2.5. *With the notation introduced above we have*:

(a) *the matrix C_Q is equal to the Cartan matrix C_B of B,*
(b) *the matrix D is upper triangular and* $\det D = \prod_{i=1}^t \dim_k \operatorname{End}({}_R\theta(i))$*,*
(c) *the matrix D_Q is lower triangular and* $\det D_Q = 1$*,*
(d) $C_Q = DD_Q$*,*
(e) $\underline{\dim}\, e_Q(M) = \underline{\dim}_\theta\, MD^T$ *for any $M \in \mathcal{F}(\theta)$.*

Proof. The item (a) follows directly from the equivalence $e_Q : \mathcal{F}(\theta) \to \mathcal{F}({}_B\Delta)$. Let D_i be i-row of the matrix D. By the equality (see Lemma 2.6 in [14])

$$\dim_k \operatorname{Hom}_R\big(Q(i), M\big) = \sum_{j=1}^t \big[M : \theta(j)\big] \dim_k \operatorname{Hom}_R\big(Q(i), \theta(j)\big),$$

we have $\dim_k \operatorname{Hom}_R(Q(i), M) = \sum_{j=1}^t [M : \theta(j)] d_{ij} = D_i(\underline{\dim}_\theta\, M)^T$. Hence (d) and (e) follow.

On the other hand, $D_Q(ij) = [Q(j) : \theta(i)] = 0$ for $i < j$. So D_Q is lower triangular and $\det D_Q = \prod_{i=1}^t [Q(i) : \theta(i)] = 1$, proving (c). To prove (b) we have

$$d_{ij} = \dim_k \operatorname{Hom}_R\big(Q(i), \theta(j)\big) = \dim_k \operatorname{Hom}_B\big({}_BP(i), {}_B\Delta(j)\big) = \big[{}_B\Delta(j), S(i)\big].$$

Thus $d_{ij} = 0$ for $i > j$, and so D is upper triangular. Finally, by Lemma 2.6 in [14], we have that $d_{ii} = \dim_k \operatorname{End}({}_R\theta(i))$. Hence, $\det D = \prod_{i=1}^t \dim_k \operatorname{End}({}_R\theta(i))$. $\square$

Definition 2.6. Associated to an epss $(\theta, \underline{Q}, \leqslant)$ of size t, we have a bilinear form $\langle -,- \rangle_Q$ on $\mathbb{Z}^t$, where $\langle x, y\rangle_Q := x(D_Q^{-T} D)y^T$.

Proposition 2.7. *For any $M, N \in \mathcal{F}(\theta)$, we have that*

(a) $\langle \underline{\dim}_\theta\, M, \underline{\dim}_\theta\, N\rangle_Q = \langle \underline{\dim}\, e_Q(M), \underline{\dim}\, e_Q(N)\rangle_B$*,*
(b) $\langle \underline{\dim}_\theta\, M, \underline{\dim}_\theta\, N\rangle_Q = \langle \underline{\dim}_{{}_B\Delta}\, e_Q(M), \underline{\dim}_{{}_B\Delta}\, e_Q(N)\rangle_{{}_B\Delta}$.

Proof. (a) By (a) and (d) in 2.5, we get $\langle x, y\rangle_B = x(D^{-T} D_Q^{-T})y^T$. Hence, the result follows from 2.5(e).

(b) From (a), we have $\langle \underline{\dim}_\theta \theta(i), \underline{\dim}_\theta \theta(j)\rangle_Q = \langle \underline{\dim}\, {}_B\Delta(i), \underline{\dim}\, {}_B\Delta(j)\rangle_B$. Then, $\langle \underline{\dim}_\theta \theta(i), \underline{\dim}_\theta \theta(j)\rangle_Q = \sum_{l\geqslant 0}(-1)^l \dim_k \mathrm{Ext}^l_B({}_B\Delta(i), {}_B\Delta(j))$ (see 1.5), proving the result. □

Theorem 2.8. *If R is a standardly stratified algebra, then the Cartan matrix C_R is invertible. Moreover, the following equality holds for any $M, N \in \mathcal{F}({}_R\Delta)$*

$$\langle \underline{\dim}_{{}_R\Delta} M, \underline{\dim}_{{}_R\Delta} N\rangle_{{}_R\Delta} = \langle \underline{\dim}\, M, \underline{\dim}\, N\rangle_R.$$

Proof. Let $({}_R\Delta, \underline{Q}, \leqslant)$ be the epss associated to $({}_R\Delta, \leqslant)$. Since the canonical stratifying system $({}_R\Delta, \leqslant)$ is standard, we have that $Q = {}_RR$. Then $B := \mathrm{End}({}_RQ)^{\mathrm{op}} = R$; therefore, by 2.5, we get $\det C_R = \prod_{i=1}^s \dim_k \mathrm{End}({}_R\Delta(i)) \neq 0$, proving that C_R is invertible. Then, from the previous proposition, we get the result. □

3. General facts about θ-directing modules

Let R be an algebra. We recall, from C.M. Ringel in [12], the following definitions. Let $\mathcal{C}$ be a full subcategory of mod R which is closed under direct summands. A *path* in $\mathcal{C}$ is a finite sequence $(X_0, X_1, \ldots, X_m)$ of indecomposable modules in $\mathcal{C}$ such that $\mathrm{rad}(X_{i-1}, X_i) \neq 0$ for all $1 \leqslant i \leqslant m$, where $\mathrm{rad}(X_{i-1}, X_i)$ is the set of all non-invertible morphisms from X_{i-1} to X_i. We write $M \preccurlyeq_{\mathcal{C}} N$ to indicate that there is a path from M to N in $\mathcal{C}$. If $m \geqslant 1$ and $X_0 \simeq X_m$, then the path $(X_0, X_1, \ldots, X_m)$ is called a *cycle* in $\mathcal{C}$. An indecomposable module X in $\mathcal{C}$ is called *$\mathcal{C}$-directing* if X does not occur in a cycle in $\mathcal{C}$. Furthermore, the category $\mathcal{C}$ is said to be *$\mathcal{C}$-directing* if any $X \in \mathrm{ind}\,\mathcal{C}$ is $\mathcal{C}$-directing. Finally, we say that $\mathcal{C}$ is *directing* if X is mod R-directing for any $X \in \mathrm{ind}\,\mathcal{C}$. Observe that, if $\mathcal{C}$ is directing then $\mathcal{C}$ is $\mathcal{C}$-directing, but the converse is false.

Let $(\theta, \preccurlyeq)$ be a stratifying system of size t. Since the category $\mathcal{F}(\theta)$ is closed under extensions and direct summands (see [13]), we can set $\mathcal{C} = \mathcal{F}(\theta)$ in the definitions above. To make simple, we replace the expression "$\mathcal{F}(\theta)$" by "θ." Thus, we get the definition of $M \preccurlyeq_\theta N$ and θ-directing. On the other hand, we say that $M \in \mathcal{F}(\theta)$ is *θ-omnipresent* if $[M : \theta(i)] \neq 0$ for any i.

Definition 3.1. For any $M \in \mathrm{ind}\,\mathcal{F}(\theta)$, we set $(\preccurlyeq_\theta, M) := \{X \in \mathrm{ind}\,\mathcal{F}(\theta)\colon\ X \preccurlyeq_\theta M\}$ and $(\preccurlyeq_\theta, M] := (\preccurlyeq_\theta, M) \cup \{M\}$. Similarly, we define $(M, \preccurlyeq_\theta) := \{X \in \mathrm{ind}\,\mathcal{F}(\theta)\colon\ X \succcurlyeq_\theta M\}$ and $[M, \preccurlyeq_\theta) := (M, \preccurlyeq_\theta) \cup \{M\}$.

Lemma 3.2. *Let $X, N \in \mathrm{ind}\,\mathcal{F}(\theta)$. If $\mathrm{Ext}^1_R(X, N) \neq 0$ then $N \preccurlyeq_\theta X$.*

Proof. Assume that $\mathrm{Ext}^1_R(X, N) \neq 0$. Consider a non-split exact sequence

$$0 \to N \xrightarrow{g} E \xrightarrow{f} X \to 0. \tag{1}$$

Let E' be an indecomposable direct summand of E. We denote by $\pi : E \to E'$ and $\iota : E' \to E$, respectively, the canonical projection and inclusion. Since the sequence (1) is non-split, we have that $g \in \mathrm{rad}(N, E)$ and $f \in \mathrm{rad}(E, X)$. Hence $\pi g \in \mathrm{rad}(N, E')$ and $f\iota \in \mathrm{rad}(E', X)$.

We assert that $\pi g \neq 0$. Indeed, suppose that $\pi g = 0$, so there exists $\varphi : X \to E'$ such that $\varphi f = \pi$ and $\varphi f \imath = \pi \imath = 1_{E'}$. Then φ is a split epimorphism, and therefore it has to be an isomorphism since X is indecomposable. Thus, (1) splits giving a contradiction, proving that $\pi g \neq 0$. Likewise, it can be proven that $f \imath \neq 0$. Then we get that $N \preccurlyeq_\theta X$ since $\mathcal{F}(\theta)$ is closed under direct summands and extensions. $\square$

Corollary 3.3. *If X is θ-directing then* $\operatorname{Ext}^1_R(X, X) = 0$.

Proof. It follows from 3.2. $\square$

In the general situation of mod R, it is well known that if $[M : S(i)] \neq 0$ for some i, then $\operatorname{Hom}_R(P(i), M) \neq 0$ and $\operatorname{Hom}_R(M, I(i)) \neq 0$, where $P(i)$ is the projective cover and $I(i)$ is the injective envelope of the simple module $S(i)$. The following lemma will be used to prove a generalization of that fact for the category $\mathcal{F}(\theta)$ (see 3.5). In this case, $\theta(i)$ is a simple object of $\mathcal{F}(\theta)$, $Q(i)$ the "Ext-projective" cover and $Y(i)$ the "Ext-injective" envelope of $\theta(i)$ in $\mathcal{F}(\theta)$.

Lemma 3.4. *If $M \in \mathcal{F}(\theta)$ then*

$$\bigl[M : \theta(i)\bigr] \dim_k \operatorname{End}_R\bigl(\theta(i)\bigr) \leqslant \min\bigl\{\dim_k \operatorname{Hom}_R\bigl(Q(i), M\bigr), \dim_k \operatorname{Hom}_R\bigl(M, Y(i)\bigr)\bigr\}.$$

Proof. The result follows from the following equalities (see Lemma 2.6 in [14])

$$\dim_k \operatorname{Hom}_R\bigl(Q(i), \theta(i)\bigr) = \dim_k \operatorname{End}_R\bigl(\theta(i)\bigr) = \dim_k \operatorname{Hom}_R\bigl(\theta(i), Y(i)\bigr),$$

$$\dim_k \operatorname{Hom}_R\bigl(Q(i), M\bigr) = \sum_{j=1}^{t} \bigl[M : \theta(j)\bigr] \dim_k \operatorname{Hom}_R\bigl(Q(i), \theta(j)\bigr),$$

$$\dim_k \operatorname{Hom}_R\bigl(M, Y(i)\bigr) = \sum_{j=1}^{t} \bigl[M : \theta(j)\bigr] \dim_k \operatorname{Hom}_R\bigl(\theta(j), Y(i)\bigr). \qquad \square$$

Corollary 3.5. *If $[M : \theta(i)] \neq 0$ then $\operatorname{Hom}_R(Q(i), M)$ and $\operatorname{Hom}_R(M, Y(i))$ are non-zero.*

Proof. It follows from 3.4. $\square$

Proposition 3.6. *Let $X \in \mathcal{F}(\theta)$. Then*

(a) *there is an exact sequence $0 \to X' \to Q_0 \xrightarrow{\varepsilon_X} X \to 0$ in $\mathcal{F}(\theta)$ such that $Q_0 \in \operatorname{add} Q$, and ε_X is the right minimal $\mathcal{P}(\theta)$-approximation of X,*
(b) *if X is indecomposable and $X \notin \operatorname{add} Q$ then $\operatorname{Ker} \varepsilon_X \in \operatorname{add}(\preccurlyeq_\theta, X)$,*
(c) *if $N \in \operatorname{ind} \mathcal{F}(\theta)$ is θ-omnipresent then $\operatorname{Ker} \varepsilon_X \in \operatorname{add}(\preccurlyeq_\theta, N)$.*

Proof. If $Q(j)$ is a direct summand of Q_0, we will denote by $\pi_j : Q_0 \to Q(j)$ the canonical projection. Let Z be an indecomposable direct summand of $\operatorname{Ker} \varepsilon_X$. Since $Z \subseteq Q_0 \in \operatorname{add} Q$, we have some direct summand $Q(j)$ of Q_0 such that $\pi_j(Z) \neq 0$. Let $g_Z := \pi_j|_Z : Z \to Q(j)$. Then, we have that $0 \neq g_Z \in \operatorname{rad}(Z, Q(j))$ since $\varepsilon_X : Q_0 \to X$ is right-minimal.

(a) It is Proposition 2.10 in [14].

(b) Assume that X is indecomposable, and $X \notin \operatorname{add} Q$. Let Z be an indecomposable direct summand of $\operatorname{Ker}\varepsilon_X$. Hence, as we have seen above, we have a morphism $g_Z : Z \to Q(j)$ such that $0 \neq g_Z \in \operatorname{rad}(Z, Q(j))$. Let $\varepsilon' := \varepsilon_X|_{Q(j)}$, so we have that $0 \neq \varepsilon' \in \operatorname{rad}(Q(j), X)$ since $\varepsilon : Q_0 \to X$ is right minimal and $X \notin \operatorname{add} Q$. Therefore, we have the path $Z \xrightarrow{g_Z} Q(j) \xrightarrow{\varepsilon'} X$ in $\mathcal{F}(\theta)$, proving that $\operatorname{Ker}\varepsilon_X \in \operatorname{add}(\preccurlyeq_\theta, X)$.

(c) Let $N \in \operatorname{ind}\mathcal{F}(\theta)$ be θ-omnipresent, and let Z be an indecomposable direct summand of $\operatorname{Ker}\varepsilon_X$. Then we have a morphism $g_Z : Z \to Q(j)$ such that $0 \neq g_Z \in \operatorname{rad}(Z, Q(j))$. Using that N is θ-omnipresent, we get from 3.5 that $\operatorname{Hom}_R(Q(j), N) \neq 0$ for any j. Let $f : Q(j) \to N$ be a non-zero morphism. Thus, we get that $Z \preccurlyeq_\theta N$ since $0 \neq g \in \operatorname{rad}(Z, Q(j))$ and $f \neq 0$, proving that $\operatorname{Ker}\varepsilon_X \in \operatorname{add}(\preccurlyeq_\theta, N)$. $\square$

The following result will be used in 3.12 to prove that, under certain conditions on θ, the vector $\underline{\dim}_\theta N$ is a positive root of q_θ.

Lemma 3.7. *Let $N \in \operatorname{ind}\mathcal{F}(\theta)$ be θ-directing. Then*

(a) $\operatorname{End}({}_R N) \simeq k$,
(b) *if* $\operatorname{id}_{\operatorname{add} Q}(N) = 0$ *then* $\operatorname{id}_{(\preccurlyeq_\theta, N]}(N) = 0$,
(c) *if* $\operatorname{pd}_{\operatorname{add} Y}(N) = 0$ *then* $\operatorname{pd}_{[N, \preccurlyeq_\theta)}(N) = 0$.

Proof. (a) Since N is indecomposable and the field k is algebraically closed, we get that $\operatorname{End}({}_R N)/\operatorname{rad}\operatorname{End}({}_R N) \simeq k$. Furthermore, using that N is θ-directing, we conclude that $\operatorname{rad}\operatorname{End}({}_R N) = 0$, proving that $\operatorname{End}({}_R N) \simeq k$.

(b) Let $\operatorname{id}_{\operatorname{add} Q}(N) = 0$. We prove by induction on j that $\operatorname{Ext}^j_R(-, N)|_{(\preccurlyeq_\theta, N]} = 0$ for all $j \geqslant 1$.

Suppose there is $X \in (\preccurlyeq_\theta, N]$ such that $\operatorname{Ext}^1_R(X, N) \neq 0$. Then by 3.2, we obtain that $N \preccurlyeq_\theta X$, and so N has to be θ-directing which is a contradiction. Hence $\operatorname{Ext}^1_R(-, N)|_{(\preccurlyeq_\theta, N]} = 0$. Assume by induction that $\operatorname{Ext}^j_R(-, N)|_{(\preccurlyeq_\theta, N]} = 0$ for $j \geqslant 2$. Then we prove that $\operatorname{Ext}^{j+1}_R(-, N)|_{(\preccurlyeq_\theta, N]} = 0$. Let $X \in (\preccurlyeq_\theta, N]$, so we may assume that $X \notin \operatorname{add} Q$ since $\operatorname{id}_{\operatorname{add} Q}(N) = 0$. By 3.6, there exists an exact sequence

$$0 \to X' \to Q_0 \to X \to 0 \quad \text{with } Q_0 \in \operatorname{add} Q \quad \text{and} \quad X' \in \operatorname{add}(\preccurlyeq_\theta, N]. \tag{2}$$

Thus $\operatorname{Ext}^j_R(X', N) = 0$ for $j \geqslant 2$. Applying the functor $\operatorname{Hom}_R(-, N)$ to the exact sequence in (2), we get the following exact sequence

$$\operatorname{Ext}^j_R(Q_0, N) \to \operatorname{Ext}^j_R(X', N) \to \operatorname{Ext}^{j+1}_R(X, N) \to \operatorname{Ext}^{j+1}_R(Q_0, N).$$

Therefore, $\operatorname{Ext}^j_R(X', N) \xrightarrow{\sim} \operatorname{Ext}^{j+1}_R(X, N)$ for $j \geqslant 1$ since $\operatorname{id}_{\operatorname{add} Q}(N) = 0$. Then, $\operatorname{Ext}^{j+1}_R(X, N) = 0$ because of the equality $\operatorname{Ext}^j_R(X', N) = 0$. Consequently, we obtain the equality $\operatorname{Ext}^{j+1}_R(-, N)|_{(\preccurlyeq_\theta, N]} = 0$, proving that $\operatorname{id}_{(\preccurlyeq_\theta, N]}(N) = 0$.

(c) It is similar to (b). $\square$

Let $(\theta, \preccurlyeq)$ be a stratifying system, and $(\theta, \underline{Q}, \preccurlyeq)$ be the epss associated to $(\theta, \preccurlyeq)$ (see 1.3). We recall from [14] that $\mathcal{F}(\theta) \cap \mathcal{P}(\theta) = \operatorname{add} Q$, where $\mathcal{P}(\theta) = \{X \in \operatorname{mod} R\colon \operatorname{Ext}^1_R(X, -)|_{\mathcal{F}(\theta)} = 0\}$. Moreover, due to Corollary 1.11 in [13], we have that the category $\mathcal{F}(\theta)$ is closed under direct summands and is functorially finite. So, by [11], we get that $\mathcal{F}(\theta)$ has relative Auslander–Reiten sequences. Furthermore, if X belongs to $\operatorname{ind}\mathcal{F}(\theta)$ and $X \notin \operatorname{add} Q$, then we denote by $\tau_\theta X$ the left-hand term in the relative Auslander–Reiten sequence $0 \to \tau_\theta X \to E \to X \to 0$ in $\mathcal{F}(\theta)$. On the other hand, if $(\theta, \underline{Y}, \preccurlyeq)$ is the eiss (see 1.2) associated to $(\theta, \preccurlyeq)$, we recall that $\mathcal{F}(\theta) \cap \mathcal{I}(\theta) = \operatorname{add} Y$, where $\mathcal{I}(\theta) = \{X \in \operatorname{mod} R\colon \operatorname{Ext}^1_R(-, X)|_{\mathcal{F}(\theta)} = 0\}$ (see [13]).

The following series of lemmas and propositions are given in order to prove 3.12, which is the main result in this section.

Lemma 3.8. *Let $N \in \operatorname{ind}\mathcal{F}(\theta)$ be θ-directing and θ-omnipresent. Then, for any $M \in (\preccurlyeq_\theta, N]$, we have the following isomorphism*

$$\operatorname{Ext}^1_R(M, -)|_{\mathcal{F}(\theta)} \xrightarrow{\sim} D\operatorname{Hom}_R(-, \tau_\theta M)|_{\mathcal{F}(\theta)}.$$

Proof. Let $M \in (\preccurlyeq_\theta, N]$. We may assume that $M \notin \operatorname{add} Q$ since $\tau_\theta Q(j) = 0$ for any j, and $\operatorname{Ext}^1_R(Q, -)|_{\mathcal{F}(\theta)} = 0$. Using that $\mathcal{F}(\theta)$ has relative Auslander–Reiten sequences (see [11]), we obtain from Corollary 9.4 in [10] that

$$\operatorname{Ext}^1_R(M, Z) \xrightarrow{\sim} D\big(\operatorname{Hom}_R(Z, \tau_\theta M)/I_\theta(Z, \tau_\theta M)\big) \quad \text{for any } Z \in \mathcal{F}(\theta),$$

where $I_\theta(Z, \tau_\theta M)$ is the set of morphisms $f : Z \to \tau_\theta M$ which factor through some object of $\operatorname{add} Y$. Using that N is θ-directing and θ-omnipresent, it can be seen, by using 3.5, that $\operatorname{Hom}_R(Y, \tau_\theta M) = 0$. Thus $I_\theta(Z, \tau_\theta M) = 0$, proving the result. □

Proposition 3.9. *Let $N \in \operatorname{ind}\mathcal{F}(\theta)$ be θ-directing and θ-omnipresent.*
If $\operatorname{Ext}^2_R(Q, Y) = 0$ then

(a) $\operatorname{Ext}^2_R(M, -)|_{\mathcal{F}(\theta)} = 0$ *for any $M \in (\preccurlyeq_\theta, N]$,*
(b) $\operatorname{resdim}_{\operatorname{add} Q}(\preccurlyeq_\theta, N] \leqslant 1$.

Proof. (a) Assume that $\operatorname{Ext}^2_R(Q, Y) = 0$. By 3.8, we have that $\operatorname{Ext}^1_R(M, -)$ is right exact on $\mathcal{F}(\theta)$ since $\operatorname{Hom}_R(-, \tau_\theta M)$ is a left exact functor. We assert that

$$\operatorname{Ext}^2_R(-, Y)|_{\mathcal{F}(\theta)} = 0. \tag{3}$$

Indeed, let $M \in \mathcal{F}(\theta)$. Then by 3.6(a), we have an exact sequence $0 \to K \to Q_0 \to M \to 0$ in $\mathcal{F}(\theta)$ with $Q_0 \in \operatorname{add} Q$. Applying the functor $\operatorname{Hom}_R(-, Y)$ to this sequence, we get the exact sequence $\operatorname{Ext}^1_R(K, Y) \to \operatorname{Ext}^2_R(M, Y) \to \operatorname{Ext}^2_R(Q_0, Y)$. Then we get that $\operatorname{Ext}^2_R(M, Y) = 0$ since $\operatorname{Ext}^1_R(K, Y) = 0 = \operatorname{Ext}^2_R(Q_0, Y)$, proving that $\operatorname{Ext}^2_R(-, Y)|_{\mathcal{F}(\theta)} = 0$.

Suppose that $M \in (\preccurlyeq_\theta, N]$ and $X \in \mathcal{F}(\theta)$. Then by Lemma 1.5 in [8], we get an exact sequence $0 \to X \to Y_0 \to Z \to 0$ in $\mathcal{F}(\theta)$ with $Y_0 \in \operatorname{add} Y$. Applying the functor $\operatorname{Hom}_R(M, -)$ to this sequence and using that $\operatorname{Ext}^1_R(M, -)$ is right exact on $\mathcal{F}(\theta)$, we get the following exact sequence

$$0 \to \operatorname{Ext}^2_R(M, X) \to \operatorname{Ext}^2_R(M, Y_0).$$

Thus, by (3), we have that $\operatorname{Ext}^2_R(M, X) = 0$, proving that $\operatorname{Ext}^2_R(M, -)|_{\mathcal{F}(\theta)} = 0$.

(b) Let $M \in (\preccurlyeq_\theta, N]$. By 3.6(a), we have an exact sequence $0 \to M' \to Q_0 \to M \to 0$ with $M' \in \mathcal{F}(\theta)$ and $Q_0 \in \operatorname{add} Q$. For any $Z \in \mathcal{F}(\theta)$, we apply the functor $\operatorname{Hom}_R(-, Z)$ to this exact sequence to get the exact sequence

$$\operatorname{Ext}^1_R(Q_0, Z) \to \operatorname{Ext}^1_R(M', Z) \to \operatorname{Ext}^2_R(M, Z).$$

Since $\operatorname{Ext}^1_R(Q_0, Z) = 0 = \operatorname{Ext}^2_R(M, Z)$ for any $Z \in \mathcal{F}(\theta)$, we obtain that $\operatorname{Ext}^1_R(M', -)|_{\mathcal{F}(\theta)} = 0$. Thus $M' \in \mathcal{F}(\theta) \cap \mathcal{P}(\theta) = \operatorname{add} Q$, and so $\operatorname{resdim}_{\operatorname{add} Q} M \leqslant 1$. □

Lemma 3.10. *Let $(\theta, \preccurlyeq)$ be a stratifying system. Then*

(a) $\operatorname{pd}_{\mathcal{F}(\theta)} \mathcal{F}(\theta) = \operatorname{pd}_{\mathcal{F}(\theta)} \theta$ *and* $\operatorname{pd} \mathcal{F}(\theta) = \operatorname{pd} \theta$,
(b) *if* $\operatorname{id}_{\operatorname{add} Q}(\mathcal{F}(\theta)) = 0$ *then* $\operatorname{pd}_{\mathcal{F}(\theta)}(M) = \operatorname{resdim}_{\operatorname{add} Q}(M)$ *for any* $M \in \mathcal{F}(\theta)$.

Proof. (a) We only prove the first part of the statement, for the second one is quite similar. Let $(\theta, \preccurlyeq)$ be a stratifying system of size t. We may assume that $\operatorname{pd}_{\mathcal{F}(\theta)} \theta = m < \infty$. Let $0 \neq X \in \mathcal{F}(\theta)$ and $i := \min X$. We prove, by reverse induction on i, that $\operatorname{pd}_{\mathcal{F}(\theta)} X \leqslant m$.

If $\max X = i$ then $X \simeq \theta(i)^{m_i}$, and so $\operatorname{pd}_{\mathcal{F}(\theta)} X \leqslant m$. Assume that $\max X \succ i$. Then, by Proposition 2.9 in [14], we have an exact sequence

$$0 \to X' \to X \to \theta(i)^{m_i} \to 0 \quad \text{with } \min X' \succ i. \tag{4}$$

For $M \in \mathcal{F}(\theta)$, we apply the functor $\operatorname{Hom}_R(-, M)$ to the exact sequence in (4) to get the following exact sequence:

$$\operatorname{Ext}^j_R\big(\theta(i)^{m_i}, M\big) \to \operatorname{Ext}^j_R(X, M) \to \operatorname{Ext}^j_R(X', M) \to \operatorname{Ext}^{j+1}_R\big(\theta(i)^{m_i}, M\big).$$

Using that $\operatorname{pd}_{\mathcal{F}(\theta)} \theta = m$, we get $\operatorname{Ext}^j_R(X, M) \xrightarrow{\sim} \operatorname{Ext}^j_R(X', M)$ for all $j \geqslant m + 1$. Hence, by reverse induction and the fact that $\min X' \succ i$, we get the equality $\operatorname{Ext}^j_R(X', M) = 0$ for all $j \geqslant m + 1$. Then $\operatorname{Ext}^j_R(X, M) = 0$ for all $j \geqslant m + 1$, proving that $\operatorname{pd}_{\mathcal{F}(\theta)} X \leqslant m$.

(b) Assume that $\operatorname{id}_{\operatorname{add} Q}(\mathcal{F}(\theta)) = 0$. Since $\operatorname{pd}_{\mathcal{F}(\theta)}(\operatorname{add} Q) = 0$ and $\operatorname{add} Q \subseteq \mathcal{F}(\theta)$, we obtain, by the dual of Theorem 2.1 in [16], the equality $\operatorname{pd}_{\mathcal{F}(\theta)}(M) = \operatorname{resdim}_{\operatorname{add} Q}(M)$ for any $M \in (\operatorname{add} Q)^\wedge$. Then, the result follows since, by Corollary 2.11 in [14], we know that $\mathcal{F}(\theta) \subseteq (\operatorname{add} Q)^\wedge$. □

In [12], C.M. Ringel proved (see 2.4 (7)) that if N is a sincere and directing R-module, then $\operatorname{pd} N \leqslant 1$, $\operatorname{id} N \leqslant 1$ and $\operatorname{gl.dim} R \leqslant 2$. In the following proposition, we generalize to "relative theory" in $\mathcal{F}(\theta)$ this result.

Proposition 3.11. *Let $N \in \operatorname{ind} \mathcal{F}(\theta)$ be θ-directing and θ-omnipresent.*

(a) *If* $\operatorname{id}_{\operatorname{add} Q}(\mathcal{F}(\theta)) = 0$, *then we have the following inequalities*

$$\operatorname{id}_{\mathcal{F}(\theta)} N \leqslant 1, \qquad \operatorname{pd}_{\mathcal{F}(\theta)}(\preccurlyeq_\theta, N] \leqslant 1 \quad \textit{and} \quad \operatorname{pd}_{\mathcal{F}(\theta)} \mathcal{F}(\theta) \leqslant 2.$$

(b) *If* $\mathrm{pd}_{\mathrm{add}\,Y}(\mathcal{F}(\theta)) = 0$, *then we have the following inequalities*

$$\mathrm{pd}_{\mathcal{F}(\theta)} N \leqslant 1, \qquad \mathrm{id}_{\mathcal{F}(\theta)}[N, \preccurlyeq_\theta) \leqslant 1 \quad \textit{and} \quad \mathrm{pd}_{\mathcal{F}(\theta)} \mathcal{F}(\theta) \leqslant 2.$$

Proof. We only prove (a) since the proof of (b) can be obtained from (a) by duality.

Assume that $\mathrm{id}_{\mathrm{add}\,Q}(\mathcal{F}(\theta)) = 0$. We start by proving that $\mathrm{Ext}^j_R(-, N)|_{\mathcal{F}(\theta)} = 0$ for all $j \geqslant 2$. Indeed, let $X \in \mathcal{F}(\theta)$, and $0 \to X' \to Q_0 \to X \to 0$ be the exact sequence of 3.6(a). Hence, by 3.6(c), we have that $X' \in \mathrm{add}(\preccurlyeq_\theta, N]$. Then, from 3.7(b), we get $\mathrm{Ext}^j_R(X', N) = 0$ for any $j \geqslant 1$. Applying the functor $\mathrm{Hom}_R(-, N)$ to the above sequence, we obtain the following exact sequence

$$\mathrm{Ext}^j_R(Q_0, N) \to \mathrm{Ext}^j_R(X', N) \to \mathrm{Ext}^{j+1}_R(X, N) \to \mathrm{Ext}^{j+1}_R(Q_0, N).$$

Therefore, $\mathrm{Ext}^j_R(X', N) \xrightarrow{\sim} \mathrm{Ext}^{j+1}_R(X, N)$ for any $j \geqslant 1$ since $\mathrm{id}_{\mathrm{add}\,Q}(\mathcal{F}(\theta)) = 0$. Thus, $\mathrm{Ext}^{j+1}_R(-, N)|_{\mathcal{F}(\theta)} = 0$ for all $j \geqslant 1$, proving that $\mathrm{id}_{\mathcal{F}(\theta)} N \leqslant 1$. The next aim is to prove the inequality $\mathrm{pd}_{\mathcal{F}(\theta)}(\preccurlyeq_\theta, N] \leqslant 1$; however, it follows easily from 3.9(b) and 3.10(b).

Finally, we prove that $\mathrm{Ext}^j_R(\theta(i), -)|_{\mathcal{F}(\theta)} = 0$ for all $j \geqslant 3$, which is enough to get the result in view of 3.10(a). To do that, we fix some i and consider the canonical exact sequence (see 1.3) $0 \to K(i) \to Q(i) \to \theta(i) \to 0$. Thus, since N is θ-omnipresent, we obtain from 3.6(c) that $K(i) \in \mathrm{add}(\preccurlyeq_\theta, N]$. Due to $\mathrm{pd}_{\mathcal{F}(\theta)}(\preccurlyeq_\theta, N] \leqslant 1$, we get that $\mathrm{Ext}^j_R(K(i), -)|_{\mathcal{F}(\theta)} = 0$ for any $j \geqslant 2$.

Applying the functor $\mathrm{Hom}_R(-, N)$ to the canonical exact sequence, for any $Z \in \mathcal{F}(\theta)$, we get the exact sequence

$$\mathrm{Ext}^j_R\big(Q(i), Z\big) \to \mathrm{Ext}^j_R\big(K(i), Z\big) \to \mathrm{Ext}^{j+1}_R\big(\theta(i), Z\big) \to \mathrm{Ext}^{j+1}_R\big(Q(i), Z\big).$$

Thus $\mathrm{Ext}^j_R(K(i), Z) \xrightarrow{\sim} \mathrm{Ext}^{j+1}_R(\theta(i), Z)$ for all $j \geqslant 1$. Hence $\mathrm{Ext}^{j+1}_R(\theta(i), -)|_{\mathcal{F}(\theta)} = 0$ for all $j \geqslant 2$, proving that $\mathrm{pd}_{\mathcal{F}(\theta)} \theta \leqslant 2$. □

The following is the main result in this section. As an application, we get 3.13, which plays an important role in the proof of the main results in Section 4.

Theorem 3.12. *Let* $(\theta, \preccurlyeq)$ *be a stratifying system such that* $\mathrm{pd}\, Q \leqslant 1$ *or* $\mathcal{I}(\theta)$ *is a coresolving subcategory of* $\mathrm{mod}\, R$. *If there is some* $X \in \mathrm{ind}\,\mathcal{F}(\theta)$ *that is* θ*-directing and* θ*-omnipresent, then*

(a) $\max(\mathrm{pd}_{\mathcal{F}(\theta)} X, \mathrm{id}_{\mathcal{F}(\theta)} X) \leqslant 1$ *and* $\mathrm{pd}_{\mathcal{F}(\theta)} \mathcal{F}(\theta) \leqslant 2$,
(b) *if* $\chi_\theta(\underline{\dim}_\theta X) = \chi_R(\underline{\dim}\, X)$ *then* $q_\theta(\underline{\dim}_\theta X) = 1$.

Proof. Assume that $\mathrm{pd}\, Q \leqslant 1$ or $\mathcal{I}(\theta)$ is coresolving.

(a) If $\mathrm{pd}\, Q \leqslant 1$, then the hypothesis needed in 3.11(a) holds since we know that $\mathrm{Ext}^1_R(Q, -)|_{\mathcal{F}(\theta)} = 0$, and so (a) follows. Otherwise, if $\mathcal{I}(\theta)$ is coresolving, then we have by Proposition 3.8(b) in [15] that $\mathrm{Ext}^j_R(\mathcal{F}(\theta), \mathcal{I}(\theta)) = 0$ for any $j > 0$. Hence, the hypothesis needed in 3.11(b) holds, proving (a).

(b) Suppose that $\chi_\theta(\underline{\dim}_\theta X) = \chi_R(\underline{\dim}\, X)$. Then, by (a), we have that $q_\theta(\underline{\dim}_\theta X) = \chi_\theta(\underline{\dim}_\theta X)$. On the other hand, from 3.7, we get $\chi_R(\underline{\dim}\, X) = 1$, proving that $q_\theta(\underline{\dim}_\theta X) = 1$. □

Corollary 3.13. *Let R be a standardly stratified algebra. If there is an $X \in \operatorname{ind}\mathcal{F}({}_R\Delta)$ that is ${}_R\Delta$-directing and ${}_R\Delta$-omnipresent, then*

$$\max(\operatorname{pd} X, \operatorname{id}_{\mathcal{F}({}_R\Delta)} X) \leqslant 1, \qquad \operatorname{pd}\mathcal{F}({}_R\Delta) \leqslant 2 \quad \textit{and} \quad q_{{}_R\Delta}(\underline{\dim}_{{}_R\Delta} X) = 1.$$

Proof. Let $({}_R\Delta, \underline{Q}, \leqslant)$ be the epss associated to $({}_R\Delta, \leqslant)$. Since R is a standardly stratified algebra, we have that $Q = {}_R R$. Moreover, due to the fact that $\operatorname{pd}\mathcal{F}({}_R\Delta)$ is finite, we get from 3.10(b) that $\operatorname{pd}_{\mathcal{F}({}_R\Delta)} M = \operatorname{pd} M$ for any $M \in \mathcal{F}({}_R\Delta)$. On the other hand, by 2.8, we know that $\chi_{{}_R\Delta}(\underline{\dim}_{{}_R\Delta} X) = \chi_R(\underline{\dim}\, X)$. Then, the result is a consequence of 3.12. $\square$

4. Main results

We start this section by fixing some notation that will be used throughout. Let $(\theta, \preccurlyeq)$ be a stratifying system. In the following, we make a construction, given in [14], to reduce each module $X \in \mathcal{F}(\theta)$ to a ${}_{B_X}\Delta$-omnipresent module over a standardly stratified algebra B_X. Indeed, for any $0 \neq X \in \mathcal{F}(\theta)$, we set $\theta_X := \{\theta(i)\colon i \in \operatorname{Supp}_\theta(X)\}$, where $\operatorname{Supp}_\theta(X) := \{i\colon [X : \theta(i)] \neq 0\}$. Then, we have the induced stratifying system $(\theta_X, \preccurlyeq)$. Consequently, we have that $\mathcal{F}(\theta_X) \subseteq \mathcal{F}(\theta)$, and that X is θ_X-omnipresent. We denote by $(\theta_X, \underline{Q_X}, \preccurlyeq)$ and $(\theta_X, \underline{Y_X}, \preccurlyeq)$ the epss and the eiss associated, respectively, to $(\theta_X, \preccurlyeq)$. Also, we have that $\mathcal{F}(\theta_X) \cap \mathcal{P}(\theta_X) = \operatorname{add} Q_X$ and $\mathcal{F}(\theta_X) \cap \mathcal{I}(\theta_X) = \operatorname{add} Y_X$. Furthermore, we have that the endomorphism algebra $B_X = \operatorname{End}({}_R Q_X)^{\mathrm{op}}$ is standardly stratified, and the functor $e_{Q_X} = \operatorname{Hom}_R(Q_X, -)\colon \mathcal{F}(\theta_X) \to \mathcal{F}({}_{B_X}\Delta)$ is an equivalence of exact categories (see Theorem 3.2 in [14]). On the other hand, we have that $X' := e_{Q_X}(X)$ is ${}_{B_X}\Delta$-omnipresent. Moreover, if X is θ-directing then X' is also ${}_{B_X}\Delta$-directing.

Lemma 4.1. *Let $X \in \operatorname{ind}\mathcal{F}(\theta)$ be θ-directing and such that $\operatorname{Ext}^2_R(Q, Y_X) = 0$. Then, for any $i, j \in \operatorname{Supp}_\theta(X)$ and $\ell = 0, 1, 2$, we have the following equality*

$$\dim_k \operatorname{Ext}^\ell_R\big(\theta(i), \theta(j)\big) = \dim_k \operatorname{Ext}^\ell_{B_X}\big({}_{B_X}\Delta(i), {}_{B_X}\Delta(j)\big).$$

Proof. Let $i, j \in \operatorname{Supp}_\theta(X)$. Since the functor $e_{Q_X}\colon \mathcal{F}(\theta_X) \to \mathcal{F}({}_{B_X}\Delta)$ is an equivalence of exact categories, we obtain the equality $\dim_k \operatorname{Ext}^\ell_R(\theta(i), \theta(j)) = \dim_k \operatorname{Ext}^\ell_{B_X}({}_{B_X}\Delta(i), {}_{B_X}\Delta(j))$ for $\ell = 0, 1$. In order to prove that the equality holds for $\ell = 2$, we use the canonical exact sequence $0 \to K_X(i) \to Q_X(i) \to \theta_X(i) \to 0$ given in 1.3. Applying the functor $\operatorname{Hom}_R(-, \theta_X(j))$ to that sequence, and since $\operatorname{Ext}^1_R(Q_X(i), \theta_X(j)) = 0$, we get the following exact sequence

$$0 \to \operatorname{Ext}^1_R\big(K_X(i), \theta_X(j)\big) \to \operatorname{Ext}^2_R\big(\theta_X(i), \theta_X(j)\big) \to \operatorname{Ext}^2_R\big(Q_X(i), \theta_X(j)\big). \tag{5}$$

We assert that $\operatorname{Ext}^2_R(Q_X(i), \theta_X(j)) = 0$. Indeed, suppose that $\operatorname{Ext}^2_R(Q_X(i), \theta_X(j)) \neq 0$. Applying the functor $\operatorname{Hom}_R(Q_X(i), -)$ to the exact sequence $0 \to \theta_X(j) \to Y_X(j) \to Z_X(j) \to 0$ that is given in 1.2, we get the following exact sequence $0 \to \operatorname{Ext}^2_R(Q_X(i), \theta_X(j)) \to \operatorname{Ext}^2_R(Q_X(i), Y_X(j))$. Hence $\operatorname{Ext}^2_R(Q_X(i), Y_X(j)) \neq 0$, and so $Q_X \notin \operatorname{add} Q$ since $\operatorname{Ext}^2_R(Q, Y_X) = 0$. Due to $Q_X(i) \in \mathcal{F}(\theta)$, we get from 3.6(a) the following exact sequence

$$0 \to K \to Q_0 \xrightarrow{\varphi} Q_X(i) \to 0 \quad \text{with } Q_0 \in \operatorname{add} Q,\ K \in \mathcal{F}(\theta), \tag{6}$$

and $\varphi\colon Q_0 \to Q_X(i)$ is the right minimal $\mathcal{P}(\theta)$-approximation of $Q_X(i)$. Applying the functor $\operatorname{Hom}_R(-, Y_X(j))$ to (6) and using that $\operatorname{Ext}^1_R(Q_0, Y_X(j)) = 0$ and $\operatorname{Ext}^2_R(Q_0, Y_X(j)) = 0$, we get $\operatorname{Ext}^1_R(K, Y_X(j)) \xrightarrow{\sim} \operatorname{Ext}^2_R(Q_X(i), Y_X(j))$. Then, $\operatorname{Ext}^1_R(K, Y_X(j)) \neq 0$ since $\operatorname{Ext}^2_R(Q_X(i), Y_X(j)) \neq 0$, and so there is an indecomposable direct summand K' of K with $\operatorname{Ext}^1_R(K', Y_X(j)) \neq 0$. Hence, by 3.2 and 3.6(b), we obtain that $Y_X(j) \preccurlyeq_\theta K' \preccurlyeq_\theta Q_X(i)$. On the other hand, since X is θ_X-omnipresent, we get from 3.5 that $\operatorname{Hom}_R(X, Y_X(j)) \neq 0$ and $\operatorname{Hom}_R(Q_X(i), X) \neq 0$. Then X belongs to a cycle in $\mathcal{F}(\theta)$, contradicting that X is θ-directing. So we have that $\operatorname{Ext}^2_R(Q_X(i), \theta_X(j)) = 0$, and then by (5), we obtain the following isomorphism

$$\operatorname{Ext}^1_R\big(K_X(i), \theta_X(j)\big) \simeq \operatorname{Ext}^2_R\big(\theta_X(i), \theta_X(j)\big). \tag{7}$$

Finally, applying the functor $\operatorname{Hom}_{B_X}(-, {}_{B_X}\Delta(j))$ to the exact sequence

$$0 \to e_{Q_X}\big(K_X(i)\big) \to e_{Q_X}\big(Q_X(i)\big) \to {}_{B_X}\Delta(i) \to 0,$$

we have the following isomorphism

$$\operatorname{Ext}^2_{B_X}\big({}_{B_X}\Delta(i), {}_{B_X}\Delta(j)\big) \simeq \operatorname{Ext}^1_{B_X}\big(e_{Q_X}\big(K_X(i)\big), {}_{B_X}\Delta(j)\big). \tag{8}$$

Then the result follows by using (7), (8) and the fact that

$$\operatorname{Ext}^1_R\big(K_X(i), \theta_X(j)\big) \simeq \operatorname{Ext}^1_{B_X}\big(e_{Q_X}\big(K_X(i)\big), {}_{B_X}\Delta(j)\big). \qquad \square$$

Proposition 4.2. *Let $X \in \operatorname{ind}\mathcal{F}(\theta)$ be θ-directing and such that $\operatorname{Ext}^2_R(Q, Y_X) = 0$. Then, for any $M \in \mathcal{F}(\theta_X)$, we have the following equalities, where $M' := e_{Q_X}(M)$*

$$q_\theta(\underline{\dim}_\theta M) = q_{B_X\Delta}(\underline{\dim}_{B_X\Delta} M') = \chi_{B_X}(\underline{\dim} M') \quad \textit{and} \quad q_\theta(\underline{\dim}_\theta X) = 1.$$

Proof. Since $X' := e_{Q_X}(X)$ is ${}_{B_X}\Delta$-directing and ${}_{B_X}\Delta$-omnipresent, we have from 3.13 that $\operatorname{pd}\mathcal{F}({}_{B_X}\Delta) \leqslant 2$ and $q_{B_X\Delta}(\underline{\dim}_{B_X\Delta} X') = 1$. Therefore, by 4.1, we get the following equalities

$$q_\theta(\underline{\dim}_\theta M) = q_{B_X\Delta}(\underline{\dim}_{B_X\Delta} M') = \chi_{B_X\Delta}(\underline{\dim}_{B_X\Delta} M').$$

On the other hand, from 2.8, we obtain that

$$\chi_{B_X\Delta}(\underline{\dim}_{B_X\Delta} M') = \chi_{B_X}(\underline{\dim} M'),$$

proving the result. $\square$

Lemma 4.3. *Let $X, Z \in \operatorname{ind}\mathcal{F}(\theta)$ and $\operatorname{Ext}^2_R(Q, Y_X) = 0$. If X is θ-directing and $\underline{\dim}_\theta X = \underline{\dim}_\theta Z$, then $X \simeq Z$.*

Proof. Assume that X is θ-directing and $\underline{\dim}_\theta X = \underline{\dim}_\theta Z$. Let $X' := e_{Q_X}(X)$ and $Z' := e_{Q_X}(Z)$. Thus, X' is ${}_{B_X}\Delta$-directing and ${}_{B_X}\Delta$-omnipresent. Then, by 3.13, we have that $\operatorname{pd} X' \leqslant 1$ and $\operatorname{id}_{\mathcal{F}({}_{B_X}\Delta)} X' \leqslant 1$. On the other hand, we have that $\underline{\dim}_{B_X\Delta} X' = \underline{\dim}_{B_X\Delta} Z'$ since the functor $e_{Q_X} = \operatorname{Hom}_R(Q_X, -) : \mathcal{F}(\theta_X) \to \mathcal{F}({}_{B_X}\Delta)$ is an equivalence of exact categories. Therefore, by 2.8 and 4.2, we obtain the following equalities

$$1 = \chi_{B_X\Delta}(\underline{\dim}_{B_X\Delta} X') = \langle \underline{\dim}_{B_X\Delta} X', \underline{\dim}_{B_X\Delta} Z' \rangle_{B_X\Delta} = \langle \underline{\dim} X', \underline{\dim} Z' \rangle_{B_X}.$$

Hence $1 = \langle \underline{\dim}\, X', \underline{\dim}\, Z' \rangle_{B_X}$. Moreover, applying 1.5 to this equality and using that $\operatorname{pd} X' \leqslant 1$, we get the following equality

$$1 = \dim_k \operatorname{Hom}_{B_X}(X', Z') - \dim_k \operatorname{Ext}^1_{B_X}(X', Z'),$$

proving that $\operatorname{Hom}_{B_X}(X', Z') \neq 0$. Similarly, by using that $\operatorname{id}_{\mathcal{F}({}_{B_X}\Delta)} X' \leqslant 1$, we obtain that $\operatorname{Hom}_{B_X}(Z', X') \neq 0$. Therefore, in view of the ${}_{B_X}\Delta$-directness of X', we get that $X' \simeq Z'$. Then $X \simeq Z$, proving the result. □

Corollary 4.4. *Let $(\theta, \preccurlyeq)$ be a stratifying system such that $\operatorname{Ext}^2_R(Q, Y_X) = 0$ for any $X \in \operatorname{ind} \mathcal{F}(\theta)$. If $\mathcal{F}(\theta)$ is θ-directing, then the correspondence*

$$X \mapsto \underline{\dim}_\theta\, X$$

induces an injection from $\operatorname{ind} \mathcal{F}(\theta)$ to the set of positive roots of q_θ.

Proof. Assume that $\mathcal{F}(\theta)$ is θ-directing. Then, by 4.2, we know that $\underline{\dim}_\theta\, X$ is a positive roof of q_θ for any $X \in \operatorname{ind} \mathcal{F}(\theta)$. Then, the result follows from 4.3. □

Lemma 4.5. *Let R be an algebra such that the Cartan matrix C_R of R is invertible, and $(\theta, \preccurlyeq)$ be a stratifying system. If $\operatorname{pd}_{\mathcal{F}(\theta)} \theta \leqslant 2$ and $\operatorname{pd} \theta < \infty$, then for any $M \in \mathcal{F}(\theta)$, we have the following equality*

$$\chi_R(\underline{\dim}\, M) = \dim_k \operatorname{End}({}_R M) - \dim_k \operatorname{Ext}^1_R(M, M) + \dim_k \operatorname{Ext}^2_R(M, M).$$

Proof. Assume that $\operatorname{pd}_{\mathcal{F}(\theta)} \theta \leqslant 2$ and $\operatorname{pd} \theta < \infty$. Since $\operatorname{pd}_{\mathcal{F}(\theta)} \theta \leqslant 2$, we get from 3.10(a) that $\operatorname{Ext}^i_R(M, M) = 0$ for $i \geqslant 3$. On the other hand, the fact that $\operatorname{pd} \theta < \infty$ implies that the projective dimension of $\mathcal{F}(\theta)$ is finite. Then, the result follows from 1.5. □

Lemma 4.6. *Let $(\theta, \preccurlyeq)$ be a stratifying system of size t, and $\mathcal{F}(\theta)$ be θ-directing. Then, for any non-negative $z \in \mathbb{Z}^t \setminus \{0\}$, there exists $M \in \mathcal{F}(\theta)$ such that $\underline{\dim}_\theta\, M = z$ and $\operatorname{Ext}^1_R(M, M) = 0$.*

Proof. Let $z \in \mathbb{Z}^t \setminus \{0\}$ be non-negative, so we have that $z = (z_1, \ldots, z_t)$, where $z_i \geqslant 0$ for any i and $z_{i_o} > 0$ for some index i_o. It is clear that there exists $M \in \mathcal{F}(\theta)$ with $\underline{\dim}_\theta\, M = z$ (for example $M = \bigoplus_{i=1}^t \theta(i)^{z_i}$). We choose such an M with $\dim_k \operatorname{End}({}_R M)$ smallest possible. Let $M = \bigoplus_{i=1}^s M_i^{s_i}$ with M_i indecomposable and pairwise non-isomorphic for all i. We assert that $\operatorname{Ext}^1_R(M_i, M_j) = 0$ for all $i \neq j$. Indeed, suppose that $\operatorname{Ext}^1_R(M_i, M_j) \neq 0$ for some $i \neq j$. Then we have a non-split exact sequence in $\mathcal{F}(\theta)$

$$0 \to M_j \to E \to M_i \to 0.$$

This sequence gives rise to the following non-split exact sequence

$$0 \to \left(\bigoplus_{\ell \neq i,j} M_\ell\right) \oplus M_j \to \left(\bigoplus_{\ell \neq i,j} M_\ell\right) \oplus E \to M_i \to 0.$$

Let $Z := (\bigoplus_{\ell \neq i,j} M_\ell) \oplus E$. Since $\mathcal{F}(\theta)$ is closed under direct summands and under extensions, we have that $Z \in \mathcal{F}(\theta)$ and so $\underline{\dim}_\theta Z = z$. Then, by Lemma 1 in Section 2.3 in [12], we have that $\dim_k \operatorname{End}({}_R Z) < \dim_k \operatorname{End}({}_R M)$, which contradicts the fact that $\dim_k \operatorname{End}({}_R M)$ is smallest possible. Hence $\operatorname{Ext}^1_R(M_i, M_j) = 0$ for all $i \neq j$. On the other hand, $\operatorname{Ext}^1_R(M_i, M_i) = 0$ for all i since M_i is θ-directing (see 3.3). □

The following result has an analogue for mod R, see 2.4(9) in [12], and the proof uses the results given before. As an application, on the one hand, we get the generalization of Theorem 2.7 in [2] for standardly stratified algebras; and on the other hand, the generalization of 2.4(9′) in [12] for the category $\mathcal{F}(\theta)$.

Theorem 4.7. *Let R be an algebra, and $(\theta, \preccurlyeq)$ be a stratifying system of size t such that the following conditions hold*:

(a) $\operatorname{Ext}^2_R(Q, Y_X) = 0$ *for any* $X \in \operatorname{ind} \mathcal{F}(\theta)$,
(b) *the category* $\mathcal{F}(\theta)$ *is* θ*-directing,*
(c) $\operatorname{pd}_{\mathcal{F}(\theta)} \theta \leqslant 2$ *and* $\operatorname{pd} \theta < \infty$,
(d) *the Cartan matrix* C_R *is invertible and* $\chi_\theta(\underline{\dim}_\theta X) = \chi_R(\underline{\dim} X)$ *for any* $X \in \mathcal{F}(\theta)$.

Then, the quadratic form q_θ is weakly positive, and the correspondence

$$X \mapsto \underline{\dim}_\theta X$$

induces a bijection from $\operatorname{ind} \mathcal{F}(\theta)$ *to the set of positive roots of* q_θ.

Proof. Since (c) and (d) holds, we can use 4.5 to obtain the following equality for any $M \in \mathcal{F}(\theta)$

$$q_\theta(\underline{\dim}_\theta M) = \dim_k \operatorname{End}({}_R M) - \dim_k \operatorname{Ext}^1_R(M, M) + \dim_k \operatorname{Ext}^2_R(M, M). \tag{9}$$

We start by proving that q_θ is weakly positive. Indeed, let $z \in \mathbb{Z}^t \setminus \{0\}$ be non-negative. Hence, by 4.6, there exists $M \in \mathcal{F}(\theta)$ such that $\underline{\dim}_\theta M = z$ and $\operatorname{Ext}^1_R(M, M) = 0$. Then, by (9), we obtain $q_\theta(z) = q_\theta(\underline{\dim}_\theta M) = \dim_k \operatorname{End}({}_R M) + \dim_k \operatorname{Ext}^2_R(M, M) > 0$, proving that q_θ is weakly positive.

On the other hand, from (a), (b) and 4.4, we conclude that the correspondence $X \mapsto \underline{\dim}_\theta X$ from $\operatorname{ind} \mathcal{F}(\theta)$ to the set of positive roots of q_θ is injective. Moreover, we assert that it is also surjective. Indeed, let z be a positive root of q_θ. Then by, 4.6 and (9), we get some $X \in \mathcal{F}(\theta)$ such that $z = \underline{\dim}_\theta X$ and $1 = q_\theta(z) = \dim_k \operatorname{End}({}_R X) + \dim_k \operatorname{Ext}^2_R(X, X)$. Hence $\dim_k \operatorname{End}({}_R X) = 1$, and so $X \in \operatorname{ind} \mathcal{F}(\theta)$, proving the result. □

The following two results generalize for standardly stratified algebras Theorem 2.7 in [2], which was proved for quasi-hereditary algebras.

Corollary 4.8. *Let R be a standardly stratified algebra, and $\mathcal{F}({}_R\Delta)$ be ${}_R\Delta$-directing. If* $\operatorname{pd} {}_R\Delta \leqslant 2$, *then the quadratic form $q_{{}_R\Delta}$ is weakly positive, and the correspondence*

$$X \mapsto \underline{\dim}_{{}_R\Delta} X$$

induces a bijection from $\operatorname{ind} \mathcal{F}({}_R\Delta)$ *to the set of positive roots of* $q_{{}_R\Delta}$.

Proof. In order to apply the theorem above, we just have to check that the hypothesis (a) and (d) needed in 4.7 hold. Let $({}_R\Delta, \underline{Q}, \leqslant)$ be the epss associated to $({}_R\Delta, \leqslant)$. Since R is a standardly stratified algebra, we get that $Q = {}_R R$. Therefore, the hypothesis (a) holds. Finally, by 2.8, we get that the Cartan matrix C_R is invertible and $\chi_{R\Delta}(\underline{\dim}_{R\Delta} X) = \chi_R(\underline{\dim}\, X)$ for any $X \in \mathcal{F}({}_R\Delta)$. $\square$

Theorem 4.9. *Let R be an algebra, and $(\theta, \preccurlyeq)$ be a stratifying system. If $\mathcal{F}(\theta)$ is θ-directing then* $\operatorname{ind}\mathcal{F}(\theta)$ *is finite.*

Proof. Suppose that $(\theta, \preccurlyeq)$ has size t, and assume that $\mathcal{F}(\theta)$ is θ-directing. For any $X \in \operatorname{ind}\mathcal{F}(\theta)$, we consider the induced stratifying system $(\theta_X, \preccurlyeq)$ as before. We assert that $\operatorname{ind}\mathcal{F}(\theta_X)$ is finite for any $X \in \operatorname{ind}\mathcal{F}(\theta)$. Indeed, since $e_{Q_X} : \mathcal{F}(\theta_X) \to \mathcal{F}({}_{B_X}\Delta)$ is an equivalence as exact categories, it is enough to prove that $\operatorname{ind}\mathcal{F}({}_{B_X}\Delta)$ is finite. Let $X' := e_{Q_X}(X)$. Since X is θ-directing, we have that X' is ${}_{B_X}\Delta$-directing and ${}_{B_X}\Delta$-omnipresent. Then, by 3.13, we have that $\operatorname{pd}{}_{B_X}\Delta \leqslant 2$. Therefore, we can apply 4.8 and the well-known fact: "the set of positive roots of a given weakly positive quadratic form is finite" to get that $\operatorname{ind}\mathcal{F}({}_{B_X}\Delta)$ is finite.

Consider the function $\Phi : \operatorname{ind}\mathcal{F}(\theta) \to 2^{[1,t]}$, where $\Phi(X) := \operatorname{Supp}_\theta(X)$. That Φ induces an equivalence relation $\sim$ on the set $\operatorname{ind}\mathcal{F}(\theta)$, and so this set can be partitioned into classes $[X] = \Phi^{-1}(\Phi(X))$ with $X \in \operatorname{ind}\mathcal{F}(\theta)$. The cardinal number $\operatorname{card}([X])$ of each class $[X]$ is finite since $[X] \subseteq \operatorname{ind}\mathcal{F}(\theta_X)$. On the other hand, $\operatorname{card}(\operatorname{ind}\mathcal{F}(\theta)/\sim) = \operatorname{card}(\operatorname{Im}\Phi) \leqslant 2^t$. Thus, we have proven that $\operatorname{ind}\mathcal{F}(\theta)$ is finite. $\square$

The following example shows that the converse of the above theorem does not hold.

Example 4.10. Let R be the quasi-hereditary algebra, which has the following presentation

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, shift right, "\alpha_1"'] & \underset{2}{\bullet} \arrow[l, shift right, "\beta_1"'] \arrow[r, shift right, "\alpha_2"'] & \underset{3}{\bullet} \arrow[l, shift right, "\beta_2"'] \end{tikzcd}$$

modulo the ideal I, of the path algebra, generated by the paths $\alpha_2\alpha_1$, $\beta_1\beta_2$, $\alpha_2\beta_2$, and $\alpha_1\beta_1 - \beta_2\alpha_2$. Then $\operatorname{ind}\mathcal{F}({}_R\Delta)$ is finite but not directing, see [17].

5. Examples of θ-directing

In this section, we give examples of algebras R such that $\operatorname{mod} R$ is not directing, but the category $\mathcal{F}(\theta)$ is θ-directing. Therefore, we get by 4.9 that $\mathcal{F}(\theta)$ is of finite representation type. Observe, that R is not so, since $\operatorname{mod} R$ is not directing.

Let R be an algebra, and let $\mathcal{C}$ be a component of the Auslander–Reiten quiver Γ_R of $\operatorname{mod} R$. We say that $\mathcal{C}$ is a *directing component* if $\operatorname{add}\mathcal{C}$ is directing (see at the beginning of Section 3); that is, any $X \in \mathcal{C}$ is directing in $\operatorname{mod} R$. Note that a cyclic path in Γ_R gives raise to a cycle in $\operatorname{mod} R$. However, there usually exists cycles in $\operatorname{mod} R$ consisting of modules belonging to different components of Γ_R.

Proposition 5.1. *Let R be an algebra and $(\theta, \preccurlyeq)$ be a stratifying system. If θ is contained into a directing component $\mathcal{C}$ of Γ_R and* $\operatorname{add}\mathcal{C}$ *is closed under extensions, then* $\operatorname{ind}\mathcal{F}(\theta)$ *is finite.*

Proof. Assume that θ is contained into a directing component $\mathcal{C}$ of Γ_R and $\operatorname{add}\mathcal{C}$ is closed under extensions. Due to 4.9, we need to prove that $\mathcal{F}(\theta)$ is θ-directing. To do that, it is enough to see that $\mathcal{F}(\theta) \subseteq \operatorname{add}\mathcal{C}$ since, by hypothesis, we know that $\mathcal{C}$ is directing. Indeed, let $0 \neq M \in \mathcal{F}(\theta)$. We apply induction on $\ell_\theta(M) := \sum_i [M : \theta(i)]$. If $\ell_\theta(M) = 1$ then $M \in \operatorname{add}\theta(i)$ for some i, and so $M \in \operatorname{add}\mathcal{C}$ since $\theta \subseteq \mathcal{C}$.

Suppose that $\ell_\theta(M) > 1$ and that $X \in \operatorname{add}\mathcal{C}$ if $\ell_\theta(X) < \ell_\theta(M)$. Then, by Proposition 2.9 in [14], we have an exact sequence $0 \to N \to M \to \theta(i)^{m_i} \to 0$ in $\mathcal{F}(\theta)$ such that $\ell_\theta(N) < \ell_\theta(M)$. Hence, $N \in \operatorname{add}\mathcal{C}$. Therefore $M \in \operatorname{add}\mathcal{C}$ since the category $\operatorname{add}\mathcal{C}$ is closed under extensions, proving that $\mathcal{F}(\theta) \subseteq \operatorname{add}\mathcal{C}$. This yields that $\mathcal{F}(\theta)$ is directing, and so $\mathcal{F}(\theta)$ has to be, in particular, θ-directing. □

According to 5.1, we need to find a directing component $\mathcal{C}$ of Γ_R such that $\operatorname{add}\mathcal{C}$ be closed under extensions. As we will see, the preprojective component is an example of such kind of components. Following C.M. Ringel in [12, p. 80], we recall that $\mathcal{C}$ is a *preprojective component* if $\mathcal{C}$, as a translation quiver, is preprojective. That means that $\mathcal{C}$ is a translation quiver without cyclic paths, with only finitely many τ-orbits and such that any τ-orbit contains a projective vertex. We also recall that a component $\mathcal{C}$ is *closed under predecessors* if for any indecomposable R-modules M and N such that $N \in \mathcal{C}$ and $\operatorname{Hom}_R(M, N) \neq 0$ we have that $M \in \mathcal{C}$. The following is one of the main properties of a preprojective component (see [12, p. 80]).

Proposition 5.2. [12] *Let R be an algebra, and $\mathcal{C}$ be a preprojective component of Γ_R. Then, $\mathcal{C}$ is a directing component which is standard and closed under predecessors.*

Lemma 5.3. *Let R be an algebra and $\mathcal{C}$ be a component of Γ_R. If $\mathcal{C}$ is closed under predecessors then* $\operatorname{add}\mathcal{C}$ *is closed under submodules and extensions.*

Proof. Suppose that the component $\mathcal{C}$ is closed under predecessors. We start by proving that $\mathcal{C}$ is closed under submodules. Indeed, Let $M \in \operatorname{add}\mathcal{C}$ and N be a submodule of M. We have a decomposition $N = \coprod_i N_i$ and $M = \coprod_j M_j$ into indecomposable modules. For any i, there is some index j_0 such that the composition of the projection $\pi_{j_0} : \coprod_j M_j \to M_{j_0}$ and the inclusion $\iota_i : N_i \to \coprod_j M_j$ is non-zero. Hence, $N_i \in \mathcal{C}$ since $\mathcal{C}$ is closed under predecessors, proving that $\mathcal{C}$ is closed under submodules.

Finally, we prove that $\mathcal{C}$ is closed under extensions. Let $0 \to M \to E \xrightarrow{f} N \to 0$ be an exact sequence with M and N belonging to $\operatorname{add}\mathcal{C}$. Then, there is a decomposition $E = X \coprod Y$ such that $f|_Y = 0$ and $g := f|_X : X \to N$ is right minimal. On the one hand, Y is a submodule of M, and so $Y \in \operatorname{add}\mathcal{C}$ since it is closed under submodules. On the other hand, we assert that X also belongs to $\operatorname{add}\mathcal{C}$. Indeed, let X' be a indecomposable direct summand of X. Since $g : X \to N$ is right minimal, we get that $g|_{X'} \neq 0$. Therefore, there is some indecomposable direct summand N' of N such that $\operatorname{Hom}_R(X', N') \neq 0$. Then we have that $X' \in \mathcal{C}$ since $\mathcal{C}$ is closed under predecessors, proving that $E \in \operatorname{add}\mathcal{C}$. □

Theorem 5.4. *Let R be an algebra and $(\theta, \preccurlyeq)$ be a stratifying system. If θ is contained into a preprojective component of Γ_R, then* $\operatorname{ind}\mathcal{F}(\theta)$ *is finite.*

Proof. Suppose that θ is contained into a preprojective component $\mathcal{C}$ of Γ_R. Then, by 5.2 and 5.3, we get that $\mathcal{C}$ is directing and $\operatorname{add}\mathcal{C}$ is closed under extensions. Therefore, the result follows from 5.1. □

Now, we give examples of stratifying systems belonging to a preprojective component. In order to do that, we assume that R is a *connected hereditary algebra of infinite representation type*. In that case, it is well known that R is the path k-algebra kQ, where Q is a connected, without cycles and not Dynkin quiver. Since the quiver Q has no cycles, we can label the set of vertices $Q_0 = \{1, 2, \dots, s\}$ in such a way that the standard module ${}_R\Delta(i)$ is the projective R-module $P(i)$ associated to the vertex i. So we can see, in this way, the hereditary algebra R as a quasi-hereditary algebra $(R, \leqslant)$, where $\leqslant$ is the natural total order on Q_0. We recall now some well-known facts about hereditary algebras (see, for example, in [12]). Since R is connected, we have only one preprojective component of Γ_R and will be denoted by $\mathcal{P}$.

Next, we recall the structure of the preprojective component $\mathcal{P}$. Let Q^{op} be the opposite quiver of Q and $\mathbb{Z}Q^{\mathrm{op}}$ the translation quiver associated to Q^{op}. We denote by $\mathbb{N}Q^{\mathrm{op}}$ the sub-translation quiver of $\mathbb{Z}Q^{\mathrm{op}}$ with vertices (n, i) such that $n \geqslant 0$ and $i \in Q_0$. Identifying the complete set of projective modules $\{P(i)\colon i \in Q_0\}$ with the set of vertices Q_0, we get that $\mathcal{P} = \mathbb{N}Q^{\mathrm{op}}$ since R is of infinite representation type. Informally speaking, the set of projective modules $\{P(i)\colon i \in Q_0\}$ becomes into a "sectional path" $\mathcal{P}_0$ in $\mathcal{P}$ and the preprojective component can be seen as $\bigcup_{n \geqslant 0} \tau^{-n}\mathcal{P}_0$, where τ is the Auslander–Reiten translation. Then, we set $\theta_n := \tau^{-n}\mathcal{P}_0$. That is $\theta_n(i) := \tau^{-n}P(i)$ for $i \in Q_0$. It is easy to see, that $\theta_0 = \mathcal{P}_0 = {}_R\Delta$.

We assert that, for any natural number n, the pair $(\theta_n, \leqslant)$ is a stratifying system of size s and $\mathcal{F}(\theta_n) = \operatorname{add}\theta_n$. Indeed, on the one hand, we have $\operatorname{Hom}_R(\theta_n(j), \theta_n(i)) \simeq \operatorname{Hom}_R({}_R\Delta(j), {}_R\Delta(i))$. On the other hand, by using the Auslander–Reiten's formula, we get $\operatorname{Ext}^1_R(\theta_n(j), \theta_n(i)) \simeq D\operatorname{Hom}_R(\theta_n(i), \tau\theta_n(j)) = 0$ for any $i, j \in Q_0$, proving the statement.

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